Eurographics Symposium on Rendering 2016
E. Eisemann and E. Fiume
(Guest Editors)

Volume 35 (2016), Number 4

Supplemental Material: Improving the Dwivedi Sampling Scheme

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1. Singular Eigenfunction Expansions

In this section, we will derive Singular Eigenfunction Expansions (SEF) in more detail than was possible in the main paper. The full derivation, which we follow very closely and which includes a proof of full-range completeness, can be found in [**?**].

As in the main paper, we will derive SEF for the adjoint Radiative Transfer Equation (RTE), which describes how importance Ψ is propagated from a sensor through the scene. The same derivation applies to the standard RTE that is used to describe the propagation of radiance.

SEF are a way of finding a solution to the RTE, and this is used in Dwivedi sampling to change the sampling distributions.

Ignoring time and wavelength dependencies, the full adjoint RTE is

$$\left[\frac{\mathrm{d}}{\mathrm{d}s} + \mu_t(\mathbf{x})\right] \Psi(\mathbf{x}, \omega) = \Psi^e(\mathbf{x}, \omega) + \mu_s(\mathbf{x}) \int_{4\pi} \phi(\omega', \omega) \Psi(\mathbf{x}, \omega') \, \mathrm{d}\omega'. \tag{1}$$

Here, $\Psi(\mathbf{x}, \omega)$ is the importance at position $\mathbf{x}$ in direction ω, and $\Psi^e(\mathbf{x}, \omega)$ describes sensors in terms of emitted importance. Similar to the standard RTE, $\mu_t(\mathbf{x})$ and $\mu_s(\mathbf{x})$ are the extinction and scattering coefficients, respectively, while $\phi(\omega', \omega)$ is the phase function.

To be able to derive an analytic solution for this equation, one introduces a few simplifications.

1. The medium is assumed to be homogeneous, so μ_t and μ_s are constants.
2. The geometric setup is assumed to be a one-dimensional slab with surface at depth $z = 0$. Directions in slab geometry can be represented by just the cosine of the angle to the surface normal $\langle \omega, n \rangle = \omega_z$.
3. There is no emission inside the volume. The slab surface can then be seen as a sensor, or importance emitter. This means that $\Psi^e(z, \omega_z) = 0$ and the RTE is a homogeneous differential equation.
4. The phase function is assumed to be constant, so that there is no preferred direction of scattering.

In total, the adjoint RTE in a homogeneous slab with isotropic scattering can be written as [**?**]

$$\left[\omega_z \frac{\partial}{\partial z} + \mu_t\right] \Psi(z, \omega_z) = \frac{\mu_s}{2} \int_{-1}^{1} \Psi(z, \omega_z') \, \mathrm{d}\omega_z'. \tag{2}$$

We now separate the spatial and directional variables using the ansatz

$$\Psi(z, \omega_z) = \varphi(\nu, \omega_z) \cdot e^{-\mu_t z/\nu}. \tag{3}$$

The directional part of this solution, $\varphi(\nu, \omega_z)$ is called an Eigenfunction. It is interesting to note that the spatial part is always non-negative, and so the directional part must be non-negative, as well, or importance would

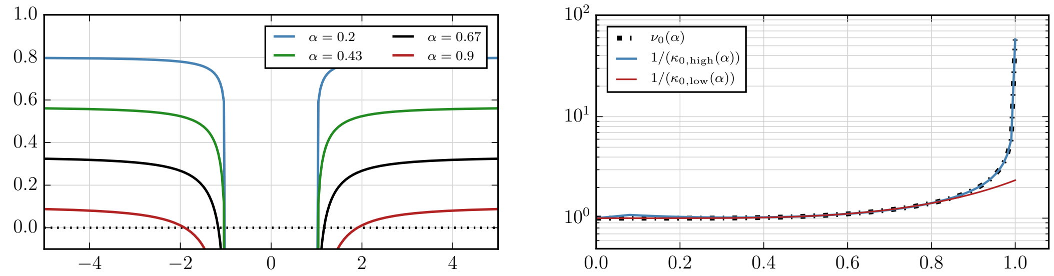

Figure 1: *Left: A plot of the left side of Eqn. (8) for varying α. The roots of this function are $\pm\nu_0$. Right: the two approximations to ν_0 for low and high albedo as given in [?]. They are overlaid on the ground truth.*

become negative. Additionally, if importance is nonzero for at least some direction ω_z, then $\varphi(\nu,\omega_z)$ must be greater than zero, which tells us that

$$\int_{-1}^{1} \varphi(\nu,\omega_z')\,\mathrm{d}\omega_z' > 0. \tag{4}$$

Substituting Eqn. (3) into Eqn. (2), we obtain

$$\begin{aligned}
\left[\omega_z \frac{\partial}{\partial z} + \mu_t\right] \varphi(\nu,\omega_z) e^{-\mu_t z/\nu} &= \frac{\mu_s}{2} e^{-\mu_t z/\nu} \int_{-1}^{1} \varphi(\nu,\omega_z')\,\mathrm{d}\omega_z' \\
\Longleftrightarrow \left[-\omega_z \frac{\mu_t}{\nu} + \mu_t\right] \varphi(\nu,\omega_z) e^{-\mu_t z/\nu} &= \frac{\mu_s}{2} e^{-\mu_t z/\nu} \int_{-1}^{1} \varphi(\nu,\omega_z')\,\mathrm{d}\omega_z' \\
\Longleftrightarrow [\nu - \omega_z]\,\varphi(\nu,\omega_z) &= \nu \frac{\alpha}{2} \int_{-1}^{1} \varphi(\nu,\omega_z')\,\mathrm{d}\omega_z',
\end{aligned} \tag{5}$$

where $\alpha = \mu_s/\mu_t$ is the single-scattering albedo. Using Eqn. (4), we can impose the normalization

$$\int_{-1}^{1} \varphi(\nu,\omega_z')\,\mathrm{d}\omega_z' = 1, \tag{6}$$

and, since φ appears on both sides of the equation, Eqn. (5) simplifies to

$$[\nu - \omega_z]\,\varphi(\nu,\omega_z) = \frac{\alpha}{2}\nu \quad \Longleftrightarrow \quad \varphi(\nu,\omega_z) = \frac{\alpha}{2}\frac{\nu}{\nu - \omega_z}. \tag{7}$$

All ν that induce valid solutions must satisfy the normalization constraint Eqn. (6). Such ν are called eigenvalues. Subsituting into the normalization constraint, we get

$$\begin{aligned}
\frac{\alpha}{2}\nu \int_{-1}^{1} \frac{1}{\nu - \omega_z}\,\mathrm{d}\omega_z &= 1 \\
\Longleftrightarrow 1 - \frac{\alpha}{2}\nu \log\left(\frac{\nu+1}{\nu-1}\right) &= 0 \\
\Longleftrightarrow 1 - \frac{\alpha}{2}\nu \log\left(\frac{1+1/\nu}{1-1/\nu}\right) &= 0 \\
\Longleftrightarrow 1 - \alpha\nu \tanh^{-1}\left(\frac{1}{\nu}\right) &= 0.
\end{aligned} \tag{8}$$

Note that these manipulations are not possible if $\nu \in [-1,1]$. As shown in Fig. 1 (left), the left side of Eqn. (8) is an even function and it has exactly two roots $\pm\nu_0$. These are called discrete eigenvalues. ν_0 is also called the diffusion length in literature [?]. We will show in Sec. 2 how to find the diffusion length in practice.

McCormick and Kuscer [?] show that there are also eigenvalues $\nu \in [-1,1]$. Those are called singular eigenvalues because $\varphi(\nu, \omega_z)$ is infinite for $\nu = \omega_z$. Full-range completeness can also be shown [**?**], and so one can write the set of all possible solutions to the RTE as the linear combination

$$\begin{aligned}\Psi(z, \omega_z) &= A(\nu_0)\varphi(\nu_0, \omega_z)e^{-\mu_t z/\nu_0} \\ &+ \int_{-1}^{1} A(\nu)\varphi(\nu, \omega_z)e^{-\mu_t z/\nu}\,\mathrm{d}\nu \\ &+ A(-\nu_0)\varphi(-\nu_0, \omega_z)e^{-\mu_t z/-\nu_0}.\end{aligned} \tag{9}$$

Here, $A(\nu)$ are coefficients. Eqn. (9) is the Singular Eigenfunction Expansion of $\Psi(z, \omega_z)$.

2. Finding the diffusion length ν_0

As shown in Sec. 1, the diffusion length ν_0 is the solution to Eqn. (8). It exists for $\alpha \in [0, 1)$ and can be found numerically using root finding algorithms. Unfortunately, these are so expensive that they cannot be used during rendering. However, Case et al. [**?**, p.55] give two series expansions for $\kappa_0 = 1/\nu_0$ that are good approximations for low and high albedo, respectively:

$$\kappa_{0,\mathrm{low}} \approx 1 - 2e^{-2/\alpha}\left(1 + \frac{4-\alpha}{\alpha}e^{-2/\alpha} + \frac{24 - 12\alpha + \alpha^2}{\alpha^2}e^{-4/\alpha} + \frac{512 - 384\alpha + 72\alpha^2 - 3\alpha^3}{\alpha^3}e^{-6/\alpha}\right) \tag{10}$$

$$\kappa_{0,\mathrm{high}} \approx \sqrt{3(1-\alpha)}\left(1 - \frac{2}{5}(1-\alpha) - \frac{12}{175}(1-\alpha)^2 - \frac{2}{125}(1-\alpha)^3 - \frac{166}{67375}(1-\alpha)^4\right) \tag{11}$$

Both approximations are overlaid on the exact solution in Fig. 1 (right). We switch between the approximations at $\alpha = 0.56$. For $\alpha = 1$, the equation cannot be satisfied, and so we cannot find ν_0. Instead, we use ν_0 computed for $\alpha = 0.9999$ whenever $\alpha \geq 0.9999$.